NONLINEAR MULTILEVEL ITERATIVE METHODS FOR MULTISCALE MODELS OF AIR/WATER FLOW IN POROUS MEDIA

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Abstract

Richards' equation and the two-phase flow equations are well-known degenerate parabolic models of air/water flow in porous media. Poor iterative solver performance and small time steps during transient simulations are often reported in field-scale simulations. In this work we study Newton-multigrid and nonlinear multigrid methods applied to discrete air/water flow models. The models are discretized using standard continuous finite element spaces. Due to strong nonlinearity and potential degeneracy in the coefficients, we stabilize the models using a multiscale approach. We present computational results comparing iterative solver performance and solution accuracy, focusing particularly on the effects of degenerate coefficients in wetting and drying problems.

1. INTRODUCTION

Two closely related models for air and water flow in non-deforming porous medium are Richards' equation and the two-phase flow equations. Both models treat the air-water-solid system as a continuum mixture, which is primarily described by the water saturation and the porous medium-scale fluid pressures [*Chavent and Jaffré*, 1986]. The models rely on nonlinear constitutive relations to reflect complex interfacial phenomena and heterogeneity that are present on smaller, unresolved scales. When the fluid mixture in the pore space degenerates, in the sense that it effectively contains only one mobile fluid phase, the nonlinear model equation may also be degenerate, in the sense that locally the linearized model equations change type from parabolic to hyperbolic or elliptic. Furthermore, even at the modeling scale, the mixing zone between two effectively single phase regions can be quite small. This combination of nonlinearity and degeneracy, and the related formation of thin boundary layers, is challenging for numerical solution techniques because the discretization must handle small-scale and potentially non-smooth solution features

while typically also requiring the solution of a large, sparse nonlinear algebraic system of equations. Since our focus in this work is mainly on the solutions of this nonlinear algebraic system, we will simplify the problem by solving only the steady state problem, which is a nonlinear elliptic boundary value problem.

Elliptic linear operators typically have an unbounded set of eigenvalues [*Evans*, 1998]. This property shows up in discrete approximations in that the finite set of eigenvalues of the discrete operator rapidly expands as the mesh is refined, which leads to rapid growth of the condition number. Classical iterative methods and Krylov subspace methods require more iterations as the condition number increases [*Kelley*, 1995] while multigrid or multilevel iterative methods have convergence rates that are independent of the grid. At the same time, the number of floating point operations required is proportional only to the number of discrete degrees of freedom. Since multilevel algorithms are also parallelizable, they are, therefore, scalable [*Briggs et al.*, 2000; *Hackbusch*, 1985; *Trottenberg et al.*, 2001]. Multilevel approaches are also remarkably flexible. The mesh-independent convergence rate has been demonstrated for variable coefficient problems with anisotropic and discontinuous coefficients, nonlinear elliptic operators, and a wide range of non-elliptic operators from computational fluid dynamics [*Hackbusch*, 1985; *Trottenberg et al.*, 2001].

In this work our objective is to study the two primary multilevel approaches for nonlinear problems: Newton-Multigrid and the full approximation scheme (FAS). Newton-Multigrid linearizes the equations first, then applies a multilevel method for the resulting linear problem. FAS places the full nonlinear problem in a multilevel setting and relies on nonlinear smoothers. A number of authors have studied multilevel iterative methods for two-phase flow and Richards' equation [*Bastian and Helmig*, 1999; *Jenkins et al.*, 2001; *Kees and Miller*, 2002; *Molenaar*, 1996; *Trangenstein*, 2002]. Our work differs from this previous work in that we consider Newton-Multigrid and FAS for the multiscale finite element discretization of a steady-state degenerate model problem with a non-smooth analytical solution.

2. METHODS

2.1. **Formulation.** We consider Richards' equation and the saturation equation of the full incompressible two-phase flow equations when written in fractional flow form. Furthermore, we limit our study to the steady state problem. The model takes the form of a nonlinear advection-diffusion equation

$$\nabla \cdot [\mathbf{f}(u) - \bar{\boldsymbol{a}}(u)\nabla\phi(u)] = 0 \tag{1}$$

where $\mathbf{f}$ is the flux vector, $\bar{\boldsymbol{a}}$ is the diffusion tensor, and ϕ is a potential, all of which may be nonlinear functions of the solution u. These quantities are derived from relations among the capillary pressure, the water and air permeabilities, the water saturation, the fluid viscosities, the fluid densities, and the gravitational acceleration [*Chavent and Jaffré*, 1986]. Given these relations there are several choices for u, $\mathbf{f}$, $\bar{\boldsymbol{a}}$ and ϕ. In particular, for common constitutive relations (e.g. [*Brooks and Corey*, 1966; *Burdine*, 1953; *Mualem*, 1976; *van Genuchten*, 1980]) and homogeneous media, u can be chosen so that $\mathbf{f}$, $\bar{\boldsymbol{a}}$, and ϕ are Lipschitz continuous functions of u and $\bar{\boldsymbol{a}}$ is a function of $\mathbf{x}$ only [*Kees*, 2004]. For Richards' equation, however, $\frac{d\phi}{du} = 0$ when $s_w = 0$, and for two-phase flow $\frac{d\phi}{du} = 0$ when $s_w = 0$ or 1. These properties cause a loss of ellipticity in the linearized operator, which

causes a loss of regularity in the solution and necessitates mesh-dependent stabilization terms in the discretization.

To focus on this type of degeneracy, we use the simple nonlinear test problem

$$\nabla \cdot [bu^q - a\nabla(u^r)] = 0 \tag{2}$$

For $r > 1$ this has roughly the same type of degeneracy as Richards' equation and two-phase flow. If we consider the one-dimensional domain $\Omega = (0, 1)$ with boundary conditions $u(0) = 1, u(1) = 0$, then we can derive analytical solutions for this equation simply by integrating it twice in space and matching the boundary conditions. We consider two special cases. For $q = r = 1$ we obtain the well known solution to the linear problem

$$u = \frac{e^{\frac{b}{a}} - e^{\frac{b}{a}x}}{e^{\frac{b}{a}} - 1} \tag{3}$$

For $q = 1, r = 2$ we obtain

$$x = \frac{2a}{b}\left[C\log\left(C - u\right) - \left(C - u\right) - D\right] \tag{4}$$

where we used Newton's method to compute C and D from the boundary conditions and to invert $x(u)$.

In general we consider a domain Ω and boundary conditions

$$u = g_D(\mathbf{x}) \quad \forall \mathbf{x} \in \Gamma_D \tag{5}$$

$$(\mathbf{f} - \bar{\boldsymbol{a}}\nabla\phi) \cdot \mathbf{n} = g_N(\mathbf{x}) \quad \forall \mathbf{x} \in \partial\Gamma_N \tag{6}$$

where $\{\Gamma_D, \Gamma_N\}$ is a partition of $\partial\Omega$. We write the weak formulation of the model equation as

$$-\int_\Omega (\mathbf{f} - \bar{\boldsymbol{a}}\nabla\phi) \cdot \nabla w d\mathbf{x} + \int_{\Gamma_N} g_N w dS = 0 \quad \forall w \in W \tag{7}$$

where W is the usual space of test functions corresponding to given boundary data.

2.2. **Discretization.** Let T_h be a partition of the domain into simplexes with maximum diameter h and take u_h, ϕ_h, and w_h to be members of the usual continuous, piecewise linear Galerkin finite element spaces. Let N_h be the nodes of the mesh. We reduce the problem to one for u_h by requiring that $\phi_h(\mathbf{x}) = u_h(\mathbf{x})$ for all $\mathbf{x} \in N_h$.

We use the multiscale framework for augmenting the standard Galerkin approach with improved accuracy and stability properties *Hughes* [1995]. Hence, we formally write the trial space for the true solution u as $V = V_h \oplus \delta V$ and and the test space as $W = W_h \oplus \delta W$. The decomposition of the test space as a direct sum yields the coupled problem for $u_h + \delta u$

$$-\int_\Omega (\mathbf{f} - \bar{\boldsymbol{a}}\nabla\phi) \cdot \nabla w_h d\mathbf{x} + \int_{\Gamma_N} g_N w_h dS = 0 \quad \forall w_h \in W_h \tag{8}$$

$$-\int_\Omega (\mathbf{f} - \bar{\boldsymbol{a}}\nabla\phi) \cdot \nabla \delta w d\mathbf{x} + \int_{\Gamma_N} g_N \delta w dS = 0 \quad \forall \delta w \in \delta W \tag{9}$$

or simply

$$F_h(u_h, \delta u) = 0 \tag{10}$$

$$F_\delta(u_h, \delta u) = 0 \tag{11}$$

Linearizing these equations about the solution $u^* = u_h^* + \delta u^*$ we obtain

$$-\int_\Omega \left(\frac{d\mathbf{f}}{du}v - \bar{\boldsymbol{a}}\frac{d\phi}{du}\nabla v\right)\cdot\nabla w_h d\mathbf{x} + \int_{\Gamma_N} g_N w_h dS = -F_h(u_h,\delta u) \quad (12)$$

$$-\int_\Omega \left(\frac{d\mathbf{f}}{du}v - \bar{\boldsymbol{a}}\frac{d\phi}{du}\nabla v\right)\cdot\nabla \delta w d\mathbf{x} + \int_{\Gamma_N} g_N \delta w dS = -F_\delta(u_h,\delta u) \quad (13)$$

where $v = v_h + \delta v$, $u_h = u_h^* + v_h$, $\delta u = \delta u^* + \delta v$. We define the bilinear form

$$a(v,w) = -\int_\Omega \left(\frac{d\mathbf{f}}{du}v - \bar{\boldsymbol{a}}\frac{d\phi}{du}\nabla v\right)\cdot\nabla w d\mathbf{x} \quad (14)$$

and denote its corresponding linear differential operator and adjoint operator by L and L^*. Using this notation and the decomposition $v = v_h + \delta v$ we write the system above as

$$a(v_h,w_h) + (\delta v, L^* w_h)_{L_2(\Omega)} = -(g_N,w_h)_{L_2(\Gamma_N)} - F_h(u_h,\delta u) \quad (15)$$

$$a(\delta v,\delta w) = -(Lv_h,\delta w)_{L_2\Omega} - (g_N,\delta w)_{L_2(\Gamma_N)} - F_\delta(u_h,\delta u) \quad (16)$$

The idea of the multiscale framework is then to approximate δv noting that the right hand side of the second equation depends on the residual of the coarse scales v_h.

We approximate the solution of the second equation in terms of v_h within each element Ω_e with [*Hughes*, 1995; *Juanes and Patzek*, 2005],

$$\delta v \approx -\tau\mathcal{R}(v_h) \approx -\tau\frac{d\mathbf{f}}{du}\cdot\nabla u \quad (17)$$

$$\tau = \left[\left(2\frac{\|\frac{d\mathbf{f}}{du}\|}{h}\right)^2 + 9\left(4\frac{\|\bar{\boldsymbol{a}}\|\frac{d\phi}{du}}{h^2}\right)^2\right]^{-1/2} \quad (18)$$

Thus we approximate the linearized problem as

$$a(v_h,w_h) - \sum_e \int_{\Omega_e} \mathcal{R}(v_h)\tau L^* w_h = -F_h(u_h+\delta u) \quad (19)$$

We then approximate the nonlinear problem as

$$F_h(u_h+\delta u) \approx \hat{F}_h(u_h) = F_h(u_h) - \sum_e \int_{\Omega_e} \mathcal{R}(u_h)\tau L^* w_h = 0 \quad (20)$$

2.3. **Iterative Methods.** We will solve the discrete problem on a hierarchical family of meshes $\{T_h\}$. For a given mesh T_{h_l} we write the next finer mesh as $T_{h_{l+1}}$. Since the finite element spaces are nested we can use the injection to map elements in V_{h_l} into $V_{h_{l+1}}$. We denote by I_l^{l+1} the matrix of the injection. The transpose of prolongation is the restriction operator I_{l+1}^l. We also consider a prolongation $\hat{I}_l^{l+1}$ which includes Dirichlet boundary conditions in its domain. Lastly we consider a second restriction operator $\tilde{I}_{l+1}^l$, which is normalized by the row sums of I_{l+1}^l. This last operator has the appropriate scaling for restricting solutions whereas I_{l+1}^l is appropriate for restricting residuals.

In this multilevel setting, we have two options for the initial iterate on each mesh. We can either take it to be the same for all grids (i.e. an interpolant of the same initial iterate) or we can use $\hat{I}_l^{l+1}$ to project the solution on the previous level. This second approach is known as nested iteration and can be applied in both linear and nonlinear contexts

We write Newton's method for $F_h(u_h) = 0$ as

$$u_h^+ = u_h^- - L_h^{-1}\hat{F}_h(u_h^-) \tag{21}$$

where L_h is the approximate Jacobian of $\hat{F}_h$ and both are defined in the previous section. This requires the solution of

$$L_h v_h = -\hat{F}_h(u_h^-) \tag{22}$$

This equation can be solved by the classical stationary iterative method damped Jacobi:

$$v^+ = v^- - \omega D^{-1}(L_h v^- - b) \tag{23}$$

where D is the diagonal of L_h.

More robust methods are Gauss-Seidel (GS) and more generally the alternating Schwarz method (ASM), which is based on a decomposition of the domain into overlapping subdomains. For this work we build subdomains by starting with the support of a nodal test function and then extending that domain to include the neighboring elements.

These three stationary iterative methods are used as smoothers in the standard full multigrid (FMG) algorithm for the prolongation and restriction (I) defined above. They can also be extended to the nonlinear problem itself:

$$u_h^+ = u_h^- - \omega D^{-1}\hat{F}_h(u_h^-) \tag{24}$$

GS and ASM are extended in the same fashion. These nonlinear iterative methods are used as nonlinear smoothers in the standard Full Approximation Scheme (FAS). FMG relies on the linearity of the problem in restricting the residual to coarser grids, approximating the error there, and then projecting the error to the fine grid. In the nonlinear setting the relation between the restricted residual and the error is only implicit. The FAS scheme uses the following nonlinear equation on the coarse grid to approximate the error v_{l-1}

$$\hat{F}_{l-1}(\tilde{I}_l^{l-1}u_l + v_{l-1}) - \hat{F}_{l-1}(\tilde{I}_l^{l-1}u_l) - I_l^{l-1}\hat{F}_l(u_l) = 0 \tag{25}$$

The coarse grid correction is then projected to level l using I_{l-1}^l. The fact that the actual solution is projected to the coarse grid is the reason we defined $\tilde{I}$; the standard restriction operator I is not scaled properly for the solution. FMG and FAS are described in more detail in [*Briggs et al.*, 2000; *Hackbusch*, 1985; *Trottenberg et al.*, 2001]. On each level we use the actual discrete operators defined in the previous section.

For any of the iterative methods above, at iteration k, we have a residual R^k (either F_h or $L_h v_h - F_h$). If requiring convergence to a specified tolerance we use the test

$$R^k < \text{rtol} * R^0 + \text{atol} \tag{26}$$

where R^0 is the initial residual.

When using an iterative method to solve equation 22, we use a safeguarded form of the Eisenstaat-Walker method to adaptively choose rtol [*Eisenstat and Walker*, 1996; *Kelley*, 1995; *Tocci et al.*, 1999]:

$$\text{rtol} = \min\left(\eta_{max}, max\left(\eta, \eta_C\right)\right) \tag{27}$$

$$\eta_C = \begin{cases} \eta_{max}, & n = 0 \\ \min\left(\eta_{max}, \eta_A\right), & n > 0, 0.1\eta_{n-1}^2 < 0.1 \\ \min\left(\eta_{max}, \max\left(\eta_A, 0.1\eta_{n-1}^2\right)\right), & n > 0, 0.1\eta_{n-1}^2 > 0.1 \end{cases} \tag{28}$$

where $\eta_A = 0.9\|F_{h,n}\|^2/\|F_{h,n-1}\|^2$, $\eta_{min} = 0.1\left(\text{rtol}_N * F_h + \text{atol}_N\right)$ and $\eta_{max} = 0.9999$

TABLE 1. Spatial error (e) and order of convergence (p)

h	0.5	0.25	0.125	0.0625	0.03125	0.015625	0.0078125
a=0.025,b=1.0,q=1,r=1							
e	3.83e-01	2.29e-01	1.19e-01	4.79e-02	1.47e-02	3.90e-03	9.91e-04
p	1.38e+00	7.46e-01	9.38e-01	1.32e+00	1.71e+00	1.91e+00	1.98e+00
a=0.025,b=1.0,q=1,r=2							
e	3.72e-01	2.31e-01	1.29e-01	6.73e-02	3.41e-02	1.71e-02	8.64e-03
p	1.43e+00	6.90e-01	8.35e-01	9.42e-01	9.82e-01	9.90e-01	9.88e-01

TABLE 2. Nested vs. non-nested Newton iterations

h	0.5	0.25	0.125	0.0625	0.03125	0.015625	0.0078125
a=0.025,b=1.0,q=1,r=2							
NI-Newton	2	2	3	3	4	4	4
Newton	2	3	4	5	6	7	8

TABLE 3. NI-Newton-GS

h	0.5	0.25	0.125	0.0625	0.03125	0.015625	0.0078125
a=0.025,b=1.0,q=1,r=1							
NI-Newton	1	2	2	2	3	3	6
GS/Newton	1	1.5	1.5	2.0	5.3	13.7	45.5
a=0.025,b=1.0,q=1,r=2							
NI-Newton	2	2	3	4	4	5	4
GS/Newton	1	1	2	3.3	8.8	13.1	99.3

3. RESULTS

We solve the two test problems on a family of seven meshes $\{T_k\} = \{T_0, \ldots, T_6\}$ with $T_0 = [0, 1/2, 1]$ and thereafter refining uniformly by bisecting the elements. First we solved the nonlinear system with Newton's method and LU factorization. In anticipation of second order spatial accuracy, we choose the nonlinear tolerances as $\text{rtol}_k = h_k^2/10$ and $\text{atol} = \min_k \text{rtol}_k$. The spatial errors (e) and rates of convergence (p) are given in table 1. The reduction in regularity of the degenerate problem ($r = 2$) is apparent from this table. For $q = r = 1$ the problem is linear and thus the Newton iteration converges in a single iteration.

In table 2 we present a table of the Newton iterations required for convergence in both a nested iteration setting (NI-Newton) and starting with the initial guess $u = 1 - x$ on all grids. This demonstrates a significant advantage for the nested iteration technique. There is some evidence that the number of Newton iterations required for convergence is nearly fixed for NI-Newton whereas Newton appears to require a growing number of iterations for the degenerate ($q = 1, r = 2$) problem.

In table 3 we show the results of replacing the direct linear solver in NI-Newton with classical stationary iterative methods. For linear iterations we present the number of iterations *per nonlinear iteration*. As is clear from the table, the residual reduction factor is growing for the stationary methods, a fact which can be demonstrated analytically for

TABLE 4. NI-Newton-FMG

h	0.5	0.25	0.125	0.0625	0.03125	0.015625	0.0078125
a=0.025,b=1.0,q=1,r=1							
NI-Newton	1	1	1	1	2	2	2
FMG-GS/Newton	1	1	1	1	1	1.5	1.5
a=0.025,b=1.0,q=1,r=2							
NI-Newton	2	2	3	3	4	4	4
FMG-GS/Newton	1	1	1	1	1	1.3	1.3

TABLE 5. FAS,a=0.025,b=1.0,q=1,r=2

h	0.5	0.25	0.125	0.0625	0.03125	0.015625	0.0078125
FAS-Jacobi	2	2	2	2	3	5	8
FAS-GS	2	1	1	2	2	3	6
FAS-ASM	2	1	1	1	1	2	3

many problems (e.g. *Trottenberg et al.* [2001]). The residual reduction factor grows from around 0.01 to 0.8. Jacobi and ASM behaved similarly.

Table 4 shows the statistics for the NI-Newton-FMG iteration. These results demonstrate the bounded (and small) residual reduction factor that is the hallmark of FMG.

In table 5 we present results for FAS with all three smoother types for the degenerate test problem ($q = 1, r = 2$).

4. CONCLUSIONS

Mesh independent convergence was attained for Newton-FMG and FAS with a variety of smoothers on two model problems. The approach uses the natural prolongation and restriction operators and a multiscale stabilized finite element formulation on each level. FAS demonstrated the most rapid convergence, but Newton-FMG also demonstrated rapid convergence while requiring only one fine grid function evaluation per iteration.

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